

# Polarity and Monopolarity of 3-colourable comparability graphs

Nikola Yolov \*  
 nikola.yolov@cs.ox.ac.uk

April 6, 2016

## Abstract

We sharpen the result that polarity and monopolarity are NP-complete problems by showing that they remain NP-complete if the input graph is restricted to be a 3-colourable comparability graph.

We start by presenting a construction reducing 1-3-SAT to monopolarity of 3-colourable comparability graphs. Then we show that polarity is at least as hard as monopolarity for input graphs restricted to a fixed disjoint-union-closed class. We conclude the paper by stating that both polarity and monopolarity of 3-colourable comparability graphs are NP-complete problems.

**Keywords:** Algorithms, Graph Theory, Complexity Theory, NP-completeness, Polar Graphs, Comparability Graphs

## 1 Introduction

A partition  $(A, B)$  of the vertices of a graph  $G$  is called *polar* if  $\overline{G[A]}$  and  $G[B]$  are unions of disjoint cliques. A polar partition  $(A, B)$  is called *monopolar* if  $A$  is an independent set, and *unipolar* if  $A$  is a clique. A graph  $G$  is called to be *polar*, *monopolar* or *unipolar* if it admits a polar, a monopolar or a unipolar partition respectively. If  $(A, B)$  is a polar partition of  $G$ , then  $(B, A)$  is polar partition of  $\overline{G}$ , hence a graph is polar iff its complement,  $\overline{G}$ , is polar. First studied by Tyshkevich and Chernyak in [TC85b] and [TC85a], monopolar graphs are a natural generalisation of bipartite and split graphs, and polar graphs are a generalisation of monopolar and co-bipartite graphs. The problems of deciding whether a graph is polar, monopolar and unipolar are called *polarity*, *monopolarity* and *unipolarity* respectively. Polarity [CC86] and monopolarity [Far04] are NP-complete problems and enjoyed a lot of attention recently. In this work we sharpen the hardness result by showing that the problems remain NP-complete even if the input graph is restricted to be a 3-colourable comparability graph. In contrast, unipolarity can be resolved in quadratic time [MY15].

---

\*The author is funded by the Engineering and Physical Sciences Research Council (EPSRC) Doctoral Training Grant and the Department of Computer Science, University of Oxford

It is shown in [LN14] that polarity and monopolarity of 3-colourable graphs are NP-complete problems. On the other hand, there are polynomial time algorithms for monopolarity and polarity of permutation graphs [EHM09], and a polynomial time algorithm for monopolarity of co-comparability graphs [CH12]. Note that a graph is a permutation graph if and only if it is both a comparability and a co-comparability graph, hence the latter algorithm is a generalisation of the prior, but less efficient. It is natural to ask whether there is a polynomial time polarity or monopolarity algorithm for the other superclass of permutation graphs – comparability graphs. In this paper we give a negative answer (provided  $P \neq NP$ ), in fact we show that even if the graph is restricted to a smaller class – the class of 3-colourable comparability graphs – the problem remains NP-hard. Polarity of comparability graphs and polarity of co-comparability graphs are easily seen to be polynomially reducible to each other, since a graph is polar iff its complement is polar, hence we can deduce that polarity of co-comparability graphs is also NP-complete. The table below summarises our discussion so far. (Here  $C$  and  $C^c$  are used to denote the classes of comparability and co-comparability graphs respectively.)

Input graph	Monopolarity	Polarity
3-colourable	NP-c [LN14]	NP-c [LN14]
$C \cap C^c$	P [EHM09]	P [EHM09]
$C^c$	P [CH12]	NP-c, this paper
$C$	NP-c, this paper	NP-c, this paper
3-colourable $\cap C$	NP-c, this paper	NP-c, this paper

We note that Churchley announced an unpublished work proving that monopolarity of comparability graphs is NP-complete. This paper is independent of it. In comparison, we address both polarity and monopolarity, and consider a smaller class – the class of 3-colourable comparability graphs, in contrast to the class of comparability graphs.

In order to show the polarity result we present a brief lemma stating that polarity is not easier than monopolarity for graph classes closed under disjoint union. The question whether there exists a class for which polarity is easier than monopolarity was asked in [LN14]. We contribute to the answer by stating that such class is certainly not closed under disjoint union.

## 2 Definitions and Notation

We use standard notation for graph theory and highlight that  $G[S]$  is used to denote the induced subgraph of  $G$  by the vertices  $S \subseteq V(G)$ . A *union of disjoint cliques* is a graph whose vertices can be partitioned into blocks, so that two vertices are joined by an edge if and only if they belong to the same block. A *co-union of disjoint cliques* is the complement of such graph.

A graph  $G$  is *k-colourable* if its vertices can be covered by  $k$  independent sets. It is NP-hard to decide whether a graph is  $k$ -colourable for  $k \geq 3$  [GJ79].

A graph  $G$  is a comparability graph if each edge can be oriented towards one of its endpoints, so that the result orientation is transitive. There is an algorithm to test if a graph is a comparability graph in  $O(\text{MM})$  time [Spi03], where  $\text{MM}$  is the time required for a matrix multiplication. A graph is *perfect* if  $\chi(H) = \omega(H)$  for every induced subgraph  $H$ . We note that there is an interesting connection between unipolar and perfect graphs – almost all perfect graphs are either unipolar or co-unipolar [PS92].

Comparability graphs are easily seen to be perfect, and therefore a 3-colourable comparability graph is simply a  $K_4$ -free comparability graph. The complete graph  $K_4$  is an example of a comparability graph which is not 3-colourable, and  $C_5$  is an example of a 3-colourable graph which is not comparability, hence 3-colourable comparability graphs is a proper subset of the classes above.

### 3 Monopolarity

In this section we show that the problem of deciding whether a 3-colourable comparability graph is monopolar is NP-complete. We call a CNF-formula  $\phi = \bigwedge_i C_i$  a **positive** 3-CNF formula, if each clause  $C_i$  contains exactly three non-negated literals. We transform the following NP-complete problem (1-3-SAT) [GJ79] into the problem above:

*Instance:* A positive 3-CNF-formula  $\phi$  with variables  $c_1 \dots c_n$ .

*Question:* Is there an assignment of the variables  $\{c_i\} \rightarrow \{true, false\}$ , such that each clause contains exactly **one** true literal.

Recall that a partition  $(A, B)$  of  $G$  is monopolar if  $A$  is an independent set and  $G[B]$  is a union of disjoint cliques. For a fixed monopolar partition  $(A, B)$ , we call a vertex  $v$  *left* if  $v \in A$ , and *right* otherwise. (Imagine that a partition  $(A, B)$  is always drawn with  $A$  on the left-hand side and  $B$  on the right-hand side). Observe that if  $v$  is a left vertex, the neighbours of  $v$  are right, hence they induce a union of disjoint cliques.

Consider the graph  $Q$  in Figure 1.

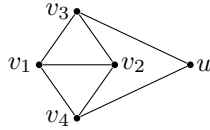


Figure 1:  $Q$

**Claim 3.1.** *The only monopolar partition of  $Q$  is  $(\{v_3, v_4\}, \{v_1, v_2, u\})$ .*

*Proof.* The neighbourhood of  $v_1$  and  $v_2$  is  $P_3$  (a path with three vertices), hence they must be both right for all monopolar partitions of  $Q$ . As  $v_3$  and  $v_4$  are not connected and they share a right neighbour, at least one of them is left.

Therefore  $u$  has a left neighbour, so  $u$  must be right. However, if  $v_1$ ,  $v_2$  and  $u$  are right, then  $v_3$  and  $v_4$  must be left.  $\square$

Consider the graph  $H$  in Figure 2. The orientation of the edges is transitive,

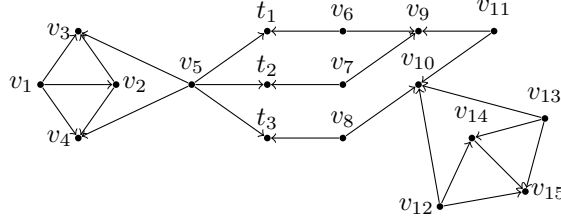


Figure 2:  $H$

hence  $H$  is a comparability graph. Further, comparability graphs are perfect, hence  $\chi(H) = \omega(H) = 3$ , i.e.  $H$  is three-colourable.

**Lemma 3.2.** *Exactly one of  $t_1$ ,  $t_2$  and  $t_3$  is right in every monopolar partition of  $H$ . There are exactly three monopolar partitions of  $H$  and each one is uniquely determined by which vertex of  $\{t_1, t_2, t_3\}$  is right.*

*Proof.* Let  $V_{Q_1} = \{v_1, \dots, v_5\}$  and  $V_{Q_2} = \{v_{10}, v_{12}, v_{13}, v_{14}, v_{15}\}$ . Observe that  $H[V_{Q_1}] \cong H[V_{Q_2}] \cong Q$ , and therefore in every monopolar partition  $v_5$  and  $v_{10}$  are right vertices and they cannot have other right neighbours from  $V_{Q_1}$  and  $V_{Q_2}$  respectively. The vertex  $v_5$  must be right, so at most one of  $\{t_1, t_2, t_3\}$  can be right. It is a routine to check that setting all  $t_1$ ,  $t_2$  and  $t_3$  left cannot yield a monopolar partition. It is also a routine to check that setting each one of  $\{t_1, t_2, t_3\}$  right and the other two left defines a unique monopolar partition.  $\square$

We associate every clause of an arbitrary positive 3-CNF formula  $\phi$  with an independent copy of  $H$ . The selection of the right vertex among  $\{t_1, t_2, t_3\}$  in a monopolar partition will indicate which of the variables of the clause is true. Then we add extra synchronisation vertices, each uniquely associated with a variable of  $\phi$  and joined by an edge to all vertices associated with the same variable in the copies of  $H$ .

More formally, let  $\phi = \bigwedge_{i=1}^m C_i$  be a positive 3-CNF formula with variables  $c_1 \dots c_n$ . Let  $G$  be the disjoint union of an independent set  $\{x_1, \dots, x_n\}$  and  $m$  disjoint copies of  $H$ ,  $\{H_i\}_{i=1}^m$ . For each clause  $C_i$ , say  $C_i = \{c_k \vee c_l \vee c_p\}$ , connect  $x_k$  with  $t_{i,1}$ ,  $x_l$  with  $t_{i,2}$  and  $x_p$  with  $t_{i,3}$ , where  $t_{i,1}$ ,  $t_{i,2}$  and  $t_{i,3}$  are respectively  $t_1$ ,  $t_2$  and  $t_3$  in  $H_i$ .

**Definition 3.3.** *For every positive 3-CNF formula  $\phi$  define  $G_\phi$  to be the graph described above.*

**Lemma 3.4.** *A positive 3-CNF formula  $\phi$  is a “yes”-instance of 1-3-SAT iff  $G_\phi$  is monopolar.*

*Proof.* ( $\Rightarrow$ ) Assume that  $f : \{c_i\} \rightarrow \{true, false\}$  is an assignment of the variables of  $\phi$  such that every clause contains exactly one true literal. Create a partition  $(A, B)$  of  $V(G_\phi)$  as follows:  $x_i \in A$  for each  $c_i$  with  $f(c_i) = true$ , and  $x_i \in B$  otherwise. The vertices  $\{x_1 \dots x_n\}$  have disjoint neighbourhoods, so we can extend the partition above with  $v \in A \Leftrightarrow x_i \in B$  for each  $x_i$  and neighbour  $v$  of  $x_i$ . By Lemma 3.2 the partition above can be further extended uniquely to each  $H_i$ , and hence the entire graph  $G_\phi$ . We conclude that  $(A, B)$  is a monopolar partition of  $G_\phi$ .

( $\Leftarrow$ ) Let  $(A, B)$  be a monopolar partition of  $G_\phi$ . Define  $f : \{c_i\} \rightarrow \{true, false\}$  as follows:  $f(c_i) = true \Leftrightarrow x_i \in A$ . From Lemma 3.2 for each clause  $C_i$  there is a unique  $t_{i,j} \in B$ , which is adjacent to  $v_{i,5} \in B$ . Observe that  $v_{i,5}$  is non-adjacent to any  $x_j$ , hence for each clause  $C$  there is a unique  $x_j \in A$  with  $c_j \in C$ .  $\square$

**Lemma 3.5.** *For every positive 3-CNF formula  $\phi$ ,  $G_\phi$  is a 3-colourable comparability graph.*

*Proof.* To see that  $G_\phi$  is a comparability graph, orient each copy of  $H$  as oriented in Figure 2. Orient the remaining edges from  $x_1 \dots x_n$  towards  $t_{i,j}$ . The described orientation of  $G_\phi$  is transitive. As  $G_\phi$  is perfect, we have  $\chi(G_\phi) = \omega(G_\phi) = 3$ .  $\square$

**Corollary 3.6.** *Monopolarity is NP-complete even if the input graph is restricted to be 3-colourable comparability graph.*

*Proof.* Monopolarity is in NP regardless of the restrictions on the input graph. Furthermore, NP-hardness follows from the reduction of 1-3-SAT in the statement of Lemma 3.4.  $\square$

It is worth noting that further restrictions can be imposed on problem. We can design  $G_\phi$  so that the different copies of  $H$  share the four triangles they contain, and therefore build  $G_\phi$  with a constant number (four) of triangles.

## 4 Polarity

In this section we prove that it is NP-complete to decide whether a 3-colourable comparability graph is polar or not. We do this by showing that polarity is at least as hard as monopolarity for classes of graphs which are closed under disjoint union. Note that 3-colourable comparability graphs form such class.

**Claim 4.1.** *Suppose  $G$  is a complement of a union of disjoint cliques. Then  $G$  is either connected or empty.*

*Proof.* Since  $\overline{G}$  is a union of disjoint cliques,  $\overline{G}$  is either a clique or disconnected. The complement of a disconnected graph is connected, therefore  $G$  is either empty or connected.  $\square$

We use the notation  $G = 2H$  to express that  $G$  is a union of two disjoint copies of  $H$  without edges inbetween.

**Lemma 4.2.** *The following three statements are equivalent:*

1.  $H = 2G$  is polar.
2.  $G$  is monopolar
3.  $H = 2G$  is monopolar.

*Proof.* (2.  $\Rightarrow$  3.) Trivial.

(3.  $\Rightarrow$  1.) Trivial.

(1.  $\Rightarrow$  2.) Let  $(A, B)$  be a polar partition of  $V(H)$ , and let  $(V_1, V_2)$  be a partition of  $V(H)$ , such that  $H[V_i] \cong G$ . Let  $A_i = A \cap V_i$ . If  $A_1$  or  $A_2$  is empty, then  $G$  is a union of cliques and hence monopolar. Otherwise,  $H[A]$  is disconnected because there are no edges between  $A_1$  and  $A_2$ . But  $H[A]$  is a co-union of cliques, hence  $H[A]$  is empty by Claim 4.1. We deduce that  $(A_i, V_i \setminus A_i)$  is a monopolar partition  $G[V_i]$ .  $\square$

**Lemma 4.3.** *Let  $\mathcal{P}$  be a class of graphs which is closed under disjoint union. Determining polarity of instances restricted to  $\mathcal{P}$  is at least as hard as determining monopolarity for the same class of instances.*

*Proof.* We reduce monopolarity for input graphs restricted to  $\mathcal{P}$  to polarity for input graphs restricted to  $\mathcal{P}$ . To decide if  $G \in \mathcal{P}$  is monopolar it is sufficient to decide whether  $2G \in \mathcal{P}$  is polar by Lemma 4.2.  $\square$

**Corollary 4.4.** *The problem of deciding whether a 3-colourable comparability graph is polar is an NP-complete problem.*

*Proof.* The problem is clearly in NP, and it is NP-complete to decide whether such graph is monopolar by Corollary 3.6, hence the statement follows from Lemma 4.3.  $\square$

**Corollary 4.5.** *It is an NP-complete problem to decide whether a co-comparability graph is polar.*

*Proof.* A graph is polar if its complement is polar. In order to decide whether a comparability graph is polar, it is sufficient to decide whether its complement, a co-comparability graph, is polar. The prior decision problem is NP-complete by Corollary 4.4, hence the latter is also NP-complete.  $\square$

## References

- [CC86] Zh Chernyak and Arkady Chernyak. About recognizing  $(\alpha, \beta)$  classes of polar. *Discrete Mathematics*, 62(2):133–138, 1986.
- [CH12] Ross Churchley and Jing Huang. Solving partition problems with colour-bipartitions. *Graphs and Combinatorics*, pages 1–12, 2012.
- [EHM09] Tınaz Ekim, Pinar Heggenes, and Daniel Meister. Polar permutation graphs. *Combinatorial Algorithms*, pages 218–229, 2009.

- [Far04] Alastair Farrugia. Vertex-partitioning into fixed additive induced-hereditary properties is NP-hard. *The Electronic Journal of Combinatorics*, 11(1):R46, 2004.
- [GJ79] Michael Garey and David Johnson. Computers and intractability: A guide to the theory of NP-completeness. *WH Freeman & Co., San Francisco*, 1979.
- [LN14] Van Bang Le and Ragnar Nevries. Complexity and algorithms for recognizing polar and monopolar graphs. *Theoretical Computer Science*, 528:1–11, 2014.
- [MY15] Colin McDiarmid and Nikola YOLOV. Recognition of unipolar and generalised split graphs. *Algorithms*, 8(1):46–59, 2015.
- [PS92] Hans Jürgen Prömel and Angelika Steger. Almost all berge graphs are perfect. *Combinatorics, Probability and Computing*, 1(01):53–79, 1992.
- [Spi03] Jeremy Spinrad. *Efficient Graph Representations.: The Fields Institute for Research in Mathematical Sciences*, volume 19. American Mathematical Society, 2003.
- [TC85a] RI Tyshkevich and AA Chernyak. Algorithms for the canonical decomposition of a graph and recognizing polarity. *Izvestia Akad. Nauk BSSR, ser. Fiz.-Mat. Nauk*, 6:16–23, 1985.
- [TC85b] RI Tyshkevich and AA Chernyak. Decomposition of graphs. *Cybernetics and Systems Analysis*, 21(2):231–242, 1985.